LINEAR ALGEBRA

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Contents

- System of linear equations
- Elementary row operations and row equivalent of matrices
- Scalar, vector and matrix
- Orthogonal vectors and orthogonal matrix
- Matrix operations
- Trace of a matrix
- Inverse of a matrix

System of linear equations

Example 1

$$\begin{cases} x_1 + 7x_2 = -57 & (1) \\ 12x_1 + 3x_2 = 45 & (2) \end{cases}$$

Step 1: $\frac{1}{3} \times (2) \Longrightarrow 4x_1 + x_2 = 15 \qquad (2*)$
Step 2: $4 \times (1) \Longrightarrow 4x_1 + 28x_2 = -228 \qquad (1*)$
Step 3: $(1*) - (2*) \Longrightarrow 27x_2 = -243$
Step 4: $x_2 = -9$
Step 5: $x_1 = 6$
Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,then we can get $\begin{bmatrix} 1 & 7 \\ 12 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -57 \\ 45 \end{bmatrix}$

$$\begin{bmatrix} 1 & 7 & \vdots & -57 \\ 12 & 3 & \vdots & 45 \end{bmatrix}$$

Elementary row operations and row equivalent of matrices

Definition 2

The following operations on the $M \times N$ matrix A, are elementary row operations:

1. multiply a row of A by a non-zero number.
2. interchange two rows.
3. replace a row by that row plus c times another row. where c is a non-zero number

Definition 3

Suppose A and B are $M \times N$ matrices. The matrix B is obtained from A by a finite sequence of elementary row operations, the B and A are equivalent.

Elementary row operations and row equivalent of matrices

Lemma 4

If A and B are equivalent, then $AX = 0$ and $BX = 0$ have the same solutions.

Proof.

Operations 1. and 2., clearly true
Let a_m be the m^{th} row of A, Suppose that row m is replaced by row m plus c times k, $c \neq 0$, $k \neq m$.
$AX = 0 \implies \sum_{n=1}^{N} a_{mn}X_n = 0$ and $\sum_{n=1}^{N} a_{kn}X_n = 0$, so that
$\sum_{n=1}^{N}(a_{mn}X_n + c \times a_{kn})X_n = \sum_{n=1}^{N} a_{mn}X_n + c \times \sum_{n=1}^{N} a_{kn}X_n = 0$.
If $BX = 0$, the $\sum_{n=1}^{N} a_{ln}X_n = 0$ for all $l \neq m$.
$0 = \sum_{n=1}^{N}(a_{mn} + c \times a_{kn})X_m = \sum_{n=1}^{N} a_{mn}X_n + c \times \sum_{n=1}^{N} a_{kn}X_n$
$= \sum_{n=1}^{N} a_{mn}X_n = 0$ □

Elementary row operations and row equivalent of matrices

Let A be an $M \times N$ matrix, X be an N dimension vector and Y be an M dimension vector. The solutions of equation $AX = Y$ can be obtained from elementary row operations from $\begin{bmatrix} A & Y \end{bmatrix}$.

Definition 5

A matrix B is row reduced if

(1) the first non-zero entry in any row is 1.

(2) each column that contains the first non-zero entry of some row has all its other entries equal to 0.

Example 6

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix} \text{ is row reduced}$$

Elementary row operations and row equivalent of matrices

Definition 7

A matrix is row reduced echelon matrix if

(1) it is row reduced

(2) any row of zeros lies below all non-zero rows

(3) if the non-zero rows are 1 through r, and the leading non-zero entry of row m is in column n_m for $m = 1 \cdots r$.

Example 8

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Elementary row operations and row equivalent of matrices

Lemma 9

Every matrix is row equivalent to a row reduced echelon matrix.

Example 10

$$\begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 6 & 8 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \\ 6 & 8 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 3 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 1 & \frac{3}{4} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

Elementary row operations and row equivalent of matrices

Remark 11

If B is row-reduced echelon matrix, the solutions of $BX = 0$ are obvious.

Example 12

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = a \Longrightarrow x_2 = -4a$ and $x_1 = -3a$.

Theorem 13

If A is an $M \times N$ matrix such that $M < N$, then $AX = 0$ has a non-zero solution.

Scalar, vector and matrix

Some concepts

1. A scalar a is single number.
2. A vertor a is $k \times 1$ list of numbers, typically arranged in a column.

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

A vector $a \in \mathbb{R}^k$(Euclidean k-dimensional space).

Scalar, vector and matrix

Some concepts

3. A matrix A is a $k \times r$ rectangular array of numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kr} \end{bmatrix} = [a_{ij}]$$

   By convention a_{ij} refers to the i-th row, j-th column of A. If $r = 1$ or $k = 1$, then A becomes a vector.

4. A matrix can be written as a set of column vectors or a set of row vectors.

   $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix}$, where $a_i = \begin{bmatrix} a_{1i} & a_{2i} & \cdots & a_{ki} \end{bmatrix}'$

   $A = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \end{bmatrix}'$, where $\alpha_j' = \begin{bmatrix} \alpha_{j1} & \alpha_{j2} & \cdots & \alpha_{jr} \end{bmatrix}$

5. A matrix is square if $k = r$.

6. A square matrix is symmetric if $A = A'$.

Scalar, vector and matrix

Some concepts

7. A square matrix is diagonal if and only if non-zero elements appear on diagonal, i.e. $a_{ij} = 0$ if $i \neq j$.
8. A square matrix is upper diagonal if all elements below the diagonal equal zero.
9. A square matrix is lower diagonal if all elements above the diagonal equal zero.
10. The transpose of a matrix, denoted $B{=}A^{'}$, is obtained by flipping the matrix on its diagonal.

$$B = A^{'} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1r} & a_{2r} & \cdots & a_{kr} \end{bmatrix}$$

Scalar, vector and matrix

Some concepts

11. A partitioned matrix

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kr} \end{bmatrix}$$

    A_{ij} denote matrices, vectors and/or scalars.

12. An important diagonal matrix is the identity matrix, which has all ones on the diagonal

$$I_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

    If A is a $k \times r$ matrix, then $AI_r = A$ and $I_k A = A$.

Scalar, vector and matrix

Matrix operations

$$A = \begin{bmatrix} a_{11} & \cdot & a_{1r} \\ \vdots & \ddots & \dots \\ a_{k1} & \cdots & a_{kr} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdot & b_{1r} \\ \vdots & \ddots & \dots \\ b_{k1} & \cdots & b_{kr} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \cdot & a_{1r} + b_{1r} \\ \vdots & \ddots & \dots \\ a_{k1} + b_{k1} & \cdots & a_{kr} + b_{kr} \end{bmatrix} \; cA = \begin{bmatrix} ca_{11} & \cdots & ca_{1r} \\ \dots & \ddots & \dots \\ ca_{k1} & \cdots & ca_{kr} \end{bmatrix}$$

A is a $k \times r$ matrix, B is an $r \times s$ matrix, then

$$[AB]_{ij} = \sum_{p=1}^{r} a_{ip} b_{pj}$$

Orthogonal vectors and orthogonalmatrix

Definition 14

Two vectors a, b are orthogonal if $a^{'}b = 0$, i.e. $\sum_{k=1}^{K} a_k b_k = 0$.

Definition 15

Suppose A is a $k \times r$ matrix, $k > r$, its columns are orthogonal if $A^{'}A = I_k$.

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix}$$

A square matrix A is orthogonal if $A^{'}A = I_k$.

Trace of matrix

Definition 16

The trace of a $k \times k$ square matrix A is the sum of its diagonal elements

$$tr(A) = \sum_{i=1}^{k} a_{ii}$$

Trace is related to the concept "degree of freedom". Suppose we regress Y on X, we can get $\hat{Y} = X\hat{\beta}$, it can be written as $\hat{Y} = S(X)Y$, where $S(X)$ is a matrix that depends X.

$$\text{"degree of freedom"} = tr(S(X))$$

$\hat{\beta} = (X'X)^{-1}X'Y$, then we can get $\hat{Y} = X(X'X)^{-1}X'Y$ and $S(X) = X(X'X)^{-1}X'$.

$$tr[X(X'X)^{-1}X'] = tr[(X'X)^{-1}X'X] = tr[I_k] = k$$

Trace of matrix

Some properties of trace

1. $tr(cA) = c \cdot tr(A)$
2. $tr(A') = tr(A)$
3. $tr(A + B) = tr(A) + tr(B)$
4. $tr(I^k) = k$
5. $tr(AB) = tr(BA)$
6. $tr(ABC) = tr(BCA) = tr(CAB)$

Inverse of a matrix

Definition 17

A $k \times k$ matrix A has full rank, or is non-singular, if there is no $c \neq 0$ such that $Ac = 0$. In this case, these exists a unique matrix B such that $AB = BA = I_k$. This matrix is called the inverse of A and denoted A^{-1}.

Some properties

(1) $AA^{-1} = A^{-1}A = I_k$

(2) $(A^{-1})' = (A')^{-1}$

(3) $(AC)^{-1} = C^{-1}A^{-1}$

Inverse of a matrix

Some properties

(4) $(A+C)^{-1} = A^{-1}(A^{-1}+C^{-1})^{-1}C^{-1}$

Proof.

$$
\begin{aligned}
(A+C)(A+C)^{-1} &= (A+C)A^{-1}(A^{-1}+C^{-1})^{-1}C^{-1} \\
&= (A+C)(C(A^{-1}+C^{-1})A)^{-1} \\
&= (A+C)((CA^{-1}+I)A)^{-1} \\
&= (A+C)(C+A)^{-1} = I
\end{aligned}
$$

□

(5) Woodbury Matrix Identity

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}$$

Special case: $U = V = I$

$$(A+C)^{-1} = A^{-1} - A^{-1}(C^{-1}+A^{-1})^{-1}A^{-1}$$

Inverse of a matrix

Proof.

$$
\begin{aligned}
&(A+UCV)(A+UCV)^{-1} \\
=&(A+UCV)[A^{-1}-A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}] \\
=&I-U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}+UCVA^{-1} \\
&-UCVA^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1} \\
=&I+UCVA^{-1}-(I+UCVA^{-1})U(C^{-1}+VA^{-1}U)^{-1}VA^{-1} \\
=&I+UCVA^{-1}-(U+UCVA^{-1}U)(C^{-1}+VA^{-1}U)^{-1}VA^{-1} \\
=&I+UCVA^{-1}-UC(C^{-1}+VA^{-1}U)(C^{-1}+VA^{-1}U)^{-1}VA^{-1} \\
=&I+UCVA^{-1}-UCVA^{-1} \\
=&I
\end{aligned}
$$

□

Inverse of a matrix

Computaton of matrix inverse

How to calculate A^{-1} ?

$$\begin{bmatrix} A & \vdots & I \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} I & \vdots & A^{-1} \end{bmatrix}$$

Example 18

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} \qquad A^{-1}?$$

Inverse of a matrix

Solution

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xRightarrow{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xRightarrow{R_3+6R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xRightarrow{R_3+4R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \xRightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xRightarrow{R_2+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xRightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \Longrightarrow \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

Inverse of a matrix

Example 19

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

Is A invertible?

Inversion of partitioned matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} & -E^{-1}BD^{-1} \\ -D^{-1}CE^{-1} & F^{-1} \end{bmatrix}$$

Where $E = A - BD^{-1}C$ and $F = D - CA^{-1}B$. E and F are Schur complement.

$$E^{-1} = (A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}BF^{-1}CA^{-1}$$
$$F^{-1} = (D - CA^{-1}B)^{-1} = D^{-1} + D^{-1}CE^{-1}BD^{-1}$$

Inverse of a matrix

Proof.

$$\begin{bmatrix} A & B & \vdots & I & 0 \\ C & D & \vdots & 0 & I \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} A & B & \vdots & I & 0 \\ D^{-1}C & I & \vdots & 0 & D^{-1} \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} A - BD^{-1}C & 0 & \vdots & I & -BD^{-1} \\ D^{-1}C & I & \vdots & 0 & D^{-1} \end{bmatrix}$$
$$\downarrow$$
$$\begin{bmatrix} I & 0 & \vdots & E^{-1} & -E^{-1}BD^{-1} \\ D^{-1}C & I & \vdots & 0 & D^{-1} \end{bmatrix}$$
$$\downarrow$$

Inverse of a matrix

Proof.

$$\left[\begin{array}{cc:cc} I & 0 & E^{-1} & -E^{-1}BD^{-1} \\ 0 & I & -D^{-1}CE^{-1} & F^{-1} \end{array}\right]$$

Then we can get

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} & -E^{-1}BD^{-1} \\ -D^{-1}EC^{-1} & F^{-1} \end{bmatrix}$$

□