LINEAR ALGEBRA

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Vector spaces

Some operations on vectors

- $1 \cdot v = v$ for all $v \in \mathbb{R}_\mathrm{N}$
- if $c_1, c_2 \in \mathbb{R}$ and $v \in \mathbb{R}_\mathrm{N}$, then $(c_1 c_2) \cdot v = c_1(c_2 v)$ for all $v \in \mathbb{R}^\mathrm{N}$ and $c_1, c_2 \in \mathbb{R}$
- $(v + w) + z = v + (w + z)$
- $c(v + w) = cv + cw$
- $(c_1 + c_2)v = c_1 v + c_2 v$

Definition 1

A vector space consists of a non-empty set V together with operations of addition and multiplication by numbers, denoted by $v + w$ and cv where v and w in V and c is a number, and these operations satisfy rules above with $\mathbb{R}^\mathrm{N}$ everywhere replaced by V.

Vector spaces

Definition 2

W is a subspace of a vector space V, if W is a subset of V and W is itself is a vector space under the operations of addition and multiplication by numbers defined on V.

$$v, w \in \mathrm{V}$$
$$av + bw \in \mathrm{V}?$$

Example 3

$\mathbb{R}^2$ is a vector space and $\{(v_1, v_2) \in \mathbb{R}^2 | v_1 + v_2 = 0$ is a subspace of $\mathbb{R}^2$.

Linear independence and bases

Definition 4

If V is a vector space, the vector $v \in \mathrm{V}$ is a linear combination of the vectors $v_1, v_2, \cdots, v_\mathrm{N}$, if there are numbers $c_1, c_2, \cdots, c_\mathrm{N}$ such that $v = c_1 v_1 + c_2 v_2 + \cdots + c_\mathrm{N} v_\mathrm{N}$.

Definition 5

If $v_1, v_2, \cdots, v_\mathrm{N} \in \mathrm{V}$, their linear span is set of all linear combinations of $v_1, v_2, \cdots, v_\mathrm{N}$. The vectors $v_1, v_2, \cdots, v_\mathrm{N}$ span V, if V is the linear span of $v_1, v_2, \cdots, v_\mathrm{N}$.

Remark 6

The linear span of $v_1, v_2, \cdots, v_\mathrm{N}$ *is a subspace of* V *and is the smallest subspace containing* $v_1, v_2, \cdots, v_\mathrm{N}$. $\mathbb{R}^2$ *is the linear span of* $(0, 1)$ *and* $(1, 0)$.

Linear independence and bases

Definition 7

The vectors $v_1, v_2, \cdots, v_N \in V$ are linearly dependent if there exist numbers $c_1, c_2, \cdots, c_N$, not all of which are zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_N v_N = 0$$

The vectors $v_1, v_2, \cdots, v_N \in V$ are linearly independent if they are not linearly dependent.

Example 8

$$(1,0,0) \quad (0,1,0) \quad (0,0,1) \quad \text{independent}$$
$$(1,0,0) \quad (0,1,0) \quad (1,1,0) \quad \text{dependent}$$

Linear independence and bases

Definition 9

A basis for a vector space V is a set of independent vectors in V that spans V.

Example 10

Let $e_n = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^{\mathrm{N}}$ where the 1 is in the n-th position. Then $e_1, e_2, \cdots, e_{\mathrm{N}}$ is the standard basis of $\mathbb{R}^{\mathrm{N}}$.

Theorem 11

If $v_1, v_2, \cdots, v_{\mathrm{M}}$ span a vector space V, then any independent set of vectors in V has no more then M elements.

Definition 12

A vector space is finite dimensional, if it has a finite basis.

Linear Independence and Bases

Definition 13

The dimension of a finite dimensional vector space V, denoted by $dim\mathrm{V}$, is the number of vectors in a basis of V.

Corollary

If V is a vector space of dimension N, then any N vectors in V that span V are independent and so are a basis of V.

Corollary

If V is a finite dimensional vector space, a basis for V is any smallest or minimal set of vectors that span V.

Linear independence and bases

Lemma 14

If $v_1, v_2, \cdots, v_M$ are independent vectors in V, and $w \in V$ does not belong to the span of $v_1, v_2, \cdots, v_M$. Then $v_1, v_2, \cdots, v_M, w$ are independent

Corollary

If V is a vector space of dimension N, then any n independent vectors in V span V and so are a basis for V.

Theorem 15

If the vectors $v_1, v_2, \cdots, v_N$ span the vector space V and $dim V > 0$, then some subset of $v_1, v_2, \cdots, v_N$ form a basis for V.

Linear independent and bases

Theorem 16

If V *is finite dimensional, non-zero vector space, any largest or maximal set of independent vectors in* V *is a basis for* V*.*

Application

This theorem suggests a way to construct a basis for a non-zero vector space V.

Theorem 17

Let W *be a non-zero subspace of a finite dimensional vector space* V *such that* $\mathrm{W} \neq \mathrm{V}$*, then* $dim\mathrm{W} < dim\mathrm{V}$*.*

Theorem 18

If $v_1, v_2, \cdots, v_{\mathrm{N}}$ *is a basis of the vector space* V*, and* $v \in \mathrm{V}$*, then the numbers* $c_1, c_2, \cdots, c_{\mathrm{N}}$ *such that* $v = \sum_{n=1}^{\mathrm{N}} c_n v_n$ *are unique.*

The row and column ranks of a matrix

Definition 19

Let A be an $M \times N$ matrix. The linear span of the rows of A is a subspace of $\mathbb{R}^N$, which is called row space. The linear span of the columns of A is a subspace of $\mathbb{R}^M$, which is called column space. The row rank of A is the dimension of the row space and the column rank of A is the dimensional of the column space.

Theorem 20

The row and column ranks of any matrix are equal.

Linear transformation

Definition 21

A and B are non-empty sets. A function $f : \mathrm{A} \to \mathrm{B}$ assigns a single point $f(a)$ in B to every point a in A.

set A: domain
set B: codomain
f: function mapping

Definition 22

Let V, W be vector spaces, then $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ is linear if for all numbers a, b and for all vectors $v_1, v_2 \in \mathrm{V}$

$$\mathrm{T}(a_1 v_1 + a_2 v_2) = a_1 \mathrm{T}(v_1) + a_2 \mathrm{T}(v_2)$$

and when $a_1 = a_2 = 0$, $\mathrm{T}(0) = 0$.

Invertible matrices and linear transformation

Theorem 23

Matrices can be used to represent linear transformation from one finite dimensional vector space to another.

Proof.

Suppose $v_1, \cdots, v_N$ is a basis for V and $w_1, \cdots, w_M$ is a basis for W. For $v \in V$, there exist $x_1, \cdots, x_N$ such that $v = \sum_{n=1}^{N} x_n v_n$ and $y_1, \cdots, y_M$ such that $T(v) = \sum_{m=1}^{M} y_m w_m$. Let $T(v_n) = \sum_{m=1}^{M} a_{mn} w_m$, then $T(v) = T(\sum_{n=1}^{N} x_n v_n) = \sum_{n=1}^{N} x_n T(v_n) = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} x_n w_n = \sum_{m=1}^{M} y_m w_m$.
Therefore we have $y_m = \sum_{n=1}^{N} a_{mn} x_n \iff y = Ax$, where $A = [a_{mn}]$. □

Invertible matrices and linear transformation

Remark 24

If the $\mathrm{M} \times \mathrm{N}$ *matrix* A *represents the linear transformation* T *and the* $\mathrm{J} \times \mathrm{M}$ *matrix* B *represents the linear transformation* S, *then the* $\mathrm{J} \times \mathrm{N}$ *matrix* BA *represents the linear transformation* $\mathrm{S} \circ \mathrm{T}$.

Remark 25

$\mathrm{N} \times \mathrm{N}$ *identity matrix* I *represents the identity function* $id_{\mathrm{V}} : \mathrm{V} \to \mathrm{V}$.

Invertible matrices and linear transformation

Definition 26

A function $f : \mathrm{V} \to \mathrm{W}$ is invertible if there exists $f^{-1} : \mathrm{W} \to \mathrm{V}$ such that $f \circ f^{-1} = id_{\mathrm{W}}$ and $f^{-1} \circ f = id_{\mathrm{V}}$, that is for all $w \in \mathrm{W}$, $f(f^{-1}(w)) = w$ and for all $v \in \mathrm{V}$, $f(f^{-1}(v)) = v$.

Definition 27

A function $f : \mathrm{V} \to \mathrm{W}$ is onto, if for all $w \in \mathrm{W}$, there exist a $v \in \mathrm{V}$ such that $f(v) = w$.

Definition 28

A function $f : \mathrm{V} \to \mathrm{W}$ is one-to-one, if for every $v_1, v_2 \in \mathrm{V}$, $\mathrm{f}(\mathrm{v}_1) \neq \mathrm{f}(\mathrm{v}_2)$ if $v_1 \neq v_2$.

Invertible matrices and linear transformation

Remark 29

$f : \mathrm{V} \to \mathrm{W}$ is invertible if and only if f is onto and one-to-one.

Theorem 30

If $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ is an invertible linear transformation. Then T^{-1} is linear.

Proof.

Let $w_1, w_2 \in \mathrm{W}$ and $v_1 = \mathrm{T}^{-1}(w_1), v_2 = \mathrm{T}^{-1}(w_2)$, then

$$\mathrm{T}(c_1 v_1 + c_2 v_2) = c_1 \mathrm{T}(v_1) + c_2 \mathrm{T}(v_2) = c_1 w_1 + c_2 w_2$$

$$c_1 \mathrm{T}^{-1}(w_1) + c_2 \mathrm{T}^{-1}(w_2) = c_1 v_1 + c_2 v_2$$
$$= T^{-1} \circ T(c_1 v_1 + c_2 v_2) = \mathrm{T}^{-1}(c_1 w_1 + c_2 w_2)$$

□

Invertible matrices and linear transformation

Theorem 31

Let $T : V \to V$ *be a linear transformation,* $v_1, v_2, \cdots, v_N$ *is a basis for* V*,* $A : N \times N$ *represents* T *with respect to* $v_1, v_2, \cdots, v_N$*. Then* T *is invertible if and only if* A *is invertible and* A^{-1} *represents* T^{-1}*.*

Proposition 32

Let $T : V \to W$ *be an invertible linear transformation,* $v_1, v_2, \cdots, v_N$ *are a basis for* V *if and only if* $T(v_1), T(v_2), \cdots, T(v_N)$ *is a basis for* W*.*

Corollary

Let $T : V \to W$ be an invertible linear transformation and V is finite dimensional, then W is finite dimensional and

$$dim W = dim V.$$

The range, rank, kernel and nullity of linear transformation

Definition 33

If $f : \mathrm{A} \to \mathrm{B}$ is a function, the range of f is $\{f(x) : x \in \mathrm{A}\}$.

Definition 34

If $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ is a linear transformation, the kernel of T is $\{v \in \mathrm{V} | \mathrm{T}(v) = 0\}$.

Theorem 35

If $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ is a linear transformation, then the range of T is a subspace of W, the kernel of T is a subspace of V.

Definition 36

If $\mathrm{T} : \mathrm{V} \to \mathrm{W}$ is a linear transformation, the rank of T is the dimension of the range of T and the nullity of T is the dimension of the kernel of T.

Theorem 37

Let $T : V \to W$ *be a linear transformation. Then* $rank\ T + nullity\ T = dim(V)$

Proof.

Let $dim(V) = N$, $v_1, v_2, \cdots, v_K$ is a basis for the null space of T, $v_1, v_2, \cdots, v_N$ is an extension of $v_1, v_2, \cdots, v_K$ to a basis for V. Then we want to show

$$T(v_{K+1}), T(v_{K+2}), \cdots T(v_N)$$

is a basis for the range of T. □

Proof.

Since $\mathrm{T}(v_1) = \cdots = \mathrm{T}(v_\mathrm{K}) = 0$, $\mathrm{T}(v_{\mathrm{K}+1}), \mathrm{T}(v_{\mathrm{K}+2}), \cdots \mathrm{T}(v_\mathrm{N})$ span the range of T. Want to show they are linearly independent. Suppose $\sum_{n=K+1}^{N} c_n \mathrm{T}(v_n) = 0$.

$$\mathrm{T}\Big(\sum_{n=\mathrm{K}+1}^{\mathrm{N}} c_n v_n\Big) = \sum_{n=K+1}^{\mathrm{N}} c_n \mathrm{T}(v_n) = 0$$

$\sum_{n=\mathrm{K}+1}^{\mathrm{N}} c_n v_n$ belongs to the kernel of T, then

$$\sum_{n=\mathrm{K}+1}^{\mathrm{N}} c_n v_n = \sum_{n=1}^{\mathrm{K}} b_n v_n$$

Therefore $\sum_{n=\mathrm{K}+1}^{\mathrm{N}} c_n v_n - \sum_{n=1}^{\mathrm{K}} b_n v_n = 0$. Since $v_1, \cdots, v_\mathrm{N}$ is a basis for V, they are independent and

$$b_1 = b_2 = \cdots = b_\mathrm{K} = 0 = c_{\mathrm{K}+1} = \cdots = c_\mathrm{N}$$

$\mathrm{T}(v_{\mathrm{K}+1}), \cdots, \mathrm{T}(v_\mathrm{N})$ are independent. □

The range, rank, kernel and nullity of linear transformation

Theorem 38

Let $T : V \to W$ *be a linear transformation and suppose* $A : M \times N$ *representing* T *with respect to the bases* $v_1, \cdots, v_N$ *and* $w_1, \cdots, w_M$. *Then the rank of* T *equals column rank of* A *and nullity of* T *equals* $N-$ *column(row) rank of* A

Singular and non-singular linear transformation

Definition 39

If $\mathrm{T} : \mathrm{V} \rightarrow \mathrm{W}$ is a linear transformation, T is non-singular if the kernel of T is $\{0\}$. T is singular if it is not non-singular, i.e. if $\mathrm{T}(v) = 0$ for some $v \neq 0$.

Remark 40

The linear transformation T is non-singular if and only if T is one-to-one.

Proof.

$$\mathrm{T}(v_1) = \mathrm{T}(v_2) \Longleftrightarrow \mathrm{T}(v_1 - v_2) = 0$$

□

Singular and non-singular linear transformation

Lemma 41

If $\mathrm{T} : \mathrm{V} \rightarrow \mathrm{W}$ *is linear transformation.* T *is non-singular if and only* $\mathrm{T}(v_1)$,$\mathrm{T}(v_2)$,$\cdots$,$\mathrm{T}(v_{\mathrm{N}})$ *are independent whenever* v_1, v_2, $\cdots$, v_{N} *are independent.*

Theorem 42

$\mathrm{T} : \mathrm{V} \rightarrow \mathrm{W}$ *is linear transformation, and* $dim(\mathrm{V}) = dim(\mathrm{W})$, *then the following are equivalent*

- T *is invertible*
- T *is non-singular*
- T *is onto*
- *if* v_1,v_2,$\cdots$,v_{N} *is a basis for* V, *then* $\mathrm{T}(v_1)$, $\mathrm{T}(v_2)$, $\cdots$, $\mathrm{T}(v_{\mathrm{N}})$ *is a basis for* W.
- *there is basis* v_1,v_2,$\cdots$,v_{N} *for* V *such that* $\mathrm{T}(v_1)$, $\mathrm{T}(v_2)$, $\cdots$, $\mathrm{T}(v_{\mathrm{N}})$ *is a basis for* W.

The inner product and norm

Definition 43

The inner product on $\mathbb{R}^{\mathrm{N}}$ is the function $< x, y >$ or $\mathrm{x} \cdot \mathrm{y}$

$$x \cdot y = \sum_{i=1}^{\mathrm{N}} x_i y_i \qquad \mathbb{R}^{\mathrm{N}} \times \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$$

Definition 44

$$||x|| = \sqrt{x \cdot x} \qquad \text{length of } x$$

Remark 45

- *if x, y are non-zero vectors in $\mathbb{R}^{\mathrm{N}}$ and if θ is the angle between x and y*
$$\cos\theta = \frac{x \cdot y}{||x|| \cdot ||y||}$$
- *Cauchy-Schwarz inequality*
$$\left|\frac{x \cdot y}{||x|| \cdot ||y||}\right| \leq 1 \Longleftrightarrow |x \cdot y| \leq ||x|| \cdot ||y||$$

Orthonormal bases and orthogonal complement

Definition 46

A set of vector $v_1, v_2, \cdots, v_M$ in $\mathbb{R}^N$ is orthogonal if $v_n \cdot v_m = 0$ if $m \neq n$.

Theorem 47

Orthogonal non-zero vectors are independent.

Definition 48

A basis $v_1, \cdots, v_M$ for a subspace V of $\mathbb{R}^N$ is orthonormal if it is orthogonal and $||v_m|| = 1$ for all m.

Theorem 49

Every non-zero subspace V of $\mathbb{R}^N$ has an orthonormal basis.(Gram-Schmidt process)

Orthonormal bases and orthogonal complement

Definition 50

If S is a subset of V, which is a subspace of $\mathbb{R}^{\mathrm{N}}$, the orthogonal complement of in V, denoted by $\mathrm{S}^{\perp}$

$$\mathrm{S}^{\perp} = \{y \in \mathrm{V} | y \cdot x = 0 \quad \text{for all} \quad x \in \mathrm{S}\}$$

Theorem 51

If W is a subspace of V and V is a subspace of $\mathbb{R}^{\mathrm{N}}$, then

$$dim(\mathrm{W}) + dim(\mathrm{W}^{\perp}) = dim(\mathrm{V})$$

Orthogonal projection

Definition 52

Let W be a subspace of V, which is a subspace of $\mathbb{R}^{\mathrm{N}}$. An orthogonal projection $\pi : \mathrm{V} \to \mathrm{W}$ is a linear transformation such that $(\mathrm{V} - \pi(\mathrm{V})) \in \mathrm{W}^{\perp}$ for all $v \in \mathrm{V}$ and $[\mathrm{V} - \pi(\mathrm{V})] \cdot \mathrm{W} = 0$ for all $w \in \mathrm{W}$.

Theorem 53

W *is subspace of* V, V *subspace of* $\mathbb{R}^{\mathrm{N}}$, *there exists a unique orthogonal projection from* V *to* W.