LINEAR ALGEBRA

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Eigenvalues and eigenvectors

Definition 1

If a non-zero vector x satisfies

$$\mathrm{A}x = \lambda x \quad \text{for some} \quad \lambda \in \mathbb{R},$$

then the vector x is called eigenvector, while the corresponding λ is called eigenvalue.

Definition 2

The characteristic equation of a square matrix A is

$$det(\mathrm{A} - \lambda \mathrm{I}) = 0$$

Eigenvalues and eigenvectors

Example 3

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$
$$|A - \lambda I| = (1 - \lambda)(3 - \lambda)$$
$$\lambda_1 = 1 \quad \lambda_2 = 3$$

Theorem 4

Suppose q_1, q_2, $\cdots$, q_m are eigenvectors of A with eigenvalues λ_1, λ_2, $\cdots$, λ_m, respectively. Assume $\lambda_i \neq \lambda_j$ if $i \neq j$. Then q_1, q_2, $\cdots$, q_m are independent.

Corollary

Let $\mathbb{L}$ be a n-dimensional linear space, and A be a linear operator in $\mathbb{L}$ with eigenvectors e_1, e_2, $\cdots$, e_n and with distinct eigenvalues λ_1, λ_2, $\cdots$, λ_n. Then e_1, e_2, $\cdots$, e_n forms a basis in $\mathbb{L}$.

Eigenvalues and eigenvectors

Some useful properties

Let λ_i and h_i, $i = 1 \cdots k$, denote the k eigenvalues and eigenvectors of a square matrix A. Let Λ be a diagonal matrix with eigenvalues in the diagonal, and let $\mathrm{H} = \begin{bmatrix} h_1 & \cdots & h_k \end{bmatrix}$.

- $det(\mathrm{A}) = \prod_{i=1}^{k} \lambda_i$.
- $tr(\mathrm{A}) = \sum_{i=1}^{k} \lambda_i$.
- A is non-singular $\iff$ all its eigenvalues are non-zero.
- If A has distinct eigenvalues, then there exist a non-singular matrix P such that $\mathrm{P}^{-1}\Lambda\mathrm{P} = \mathrm{A}$.
- If A is symmetric, its eigenvalues are all real, then $\mathrm{A} = \mathrm{H}\Lambda\mathrm{H}'$ and $\mathrm{H}'\mathrm{A}\mathrm{H} = \Lambda$.
- $\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_k^{-1}$ are the eigenvalues of A^{-1}.

Note that $\mathrm{A} = \mathrm{H}\Lambda\mathrm{H}'$ is called spectral decomposition.

Some useful matrix decomposition

Spectral decomposition

Let A be a $m \times m$ real symmetric matrix. Then there exists an orthogonal matrix P such that $\mathrm{P}'\mathrm{A}\mathrm{P} = \Lambda$ or $\mathrm{A} = \mathrm{P}\Lambda\mathrm{P}'$ where Λ is a diagonal matrix.

Singular value decomposition

For any $\mathrm{A} : m \times n$, we have

$$\mathrm{A} = \mathrm{U}\Sigma\mathrm{V}', \text{ where}$$

U is an orthogonal matrix whose columns are the eigenvectors of $\mathrm{A}\mathrm{A}'$.

V is an orthogonal matrix whose columns are the eigenvectors of $\mathrm{A}'\mathrm{A}$.

Σ is an all zero matrix except for the first r diagonal elements

$$\sigma_i = \Sigma_{ii} \quad i = 1 \cdots r$$

that are the square root of the eigenvalues of $\mathrm{A}\mathrm{A}'$ or $\mathrm{A}'\mathrm{A}$.

Some useful matrix decomposition

QR decomposition

For an $m \times n$ matrix A whose columns are linearly independent, we have

$$\mathrm{A} = \mathrm{QR}$$

where Q is an $m \times n$ matrix whose columns form an orthonormal basis for the column space of A. R is an non-singular $n \times n$ upper triangular matrix.

Example 5

If $\mathrm{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$, by Gram-Schmidt process, we have

$$u_{k+1} = a_{k+1} - a_{k+1} \cdot e_1 \cdot e_1 - \cdots a_{k+1} \cdot e_k \cdot e_k \qquad e_{k+1} = \frac{u_{k+1}}{||u_{k+1}||}$$

$$\mathrm{A} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \begin{bmatrix} a_1 e_1 & a_2 e_1 & \cdots & a_n e_1 \\ & a_2 e_2 & \cdots & a_n e_2 \\ & & \ddots & \vdots \\ & & & a_n e_n \end{bmatrix}$$

Some useful matrix decomposition

LU decomposition

Let A be an $m \times m$ non-singular matrix. Then there exist L and U such that L is a lower triangular matrix and U is a upper triangular matrix and $\mathrm{A} = \mathrm{LU}$.

$$\mathrm{A} \longrightarrow \mathrm{U}$$

$$\mathrm{E}_k \cdots \mathrm{E}_2 \mathrm{E}_1 \mathrm{A} = \mathrm{U}$$

If each such elementary matrix E_i is a lower triangular matrix, we can show that E_i^{-1} is also lower triangular.

$$\mathrm{A} = \mathrm{E}_1^{-1} \mathrm{E}_2^{-1} \cdots \mathrm{E}_k^{-1} \mathrm{U}$$

$$\mathrm{A} = \mathrm{LU}$$

- If A is non-singular, for each L, the upper triangular matrix U is unique.
- But an LU decomposition is not unique.
- To find out the unique LU decomposition, it is necessary to put some restrictions on L and U. For example, all diagonal entries of L are 1.

Some useful matrix decomposition

Cholesky decomposition

If A is an $n \times n$ real symmetric positive definite matrix, then there exists a unique lower triangular matrix G with positive diagonal elements such that $\mathrm{A} = \mathrm{G} \cdot \mathrm{G}'$